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Simultaneous Design of Active Vibration Control and Passive Viscous Damping

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Structural engineers have found that passive damping can reduce the amount of active damping required to control structural vibration. Conversely, improperly designed passive damping can inadvertently increase system reaction times, reducing control effectiveness. This paper presents several techniques for blending active vibration control and passive viscous damping. A closed-form optimal solution for minimizing a quadratic cost functional is derived, but it is shown to be dependent on the initial conditions and produces time-varying damping coefficients. To eliminate the dependence on initial conditions, solution techniques for suboptimal, state independent solutions are developed. The suboptimal solutions require less computation effort, but still give good estimates of the optimal solution. The advantages and disadvantages of the different solution techniques are discussed with respect to computation requirements and robustness. Methods of comparing competing designs are also discussed. Several numerical examples illustrate the similarities and differences of the various techniques. More importantly, the examples demonstrate the significant improvements achievable by simultaneously designing passive and active damping.

Introduction

TRADITIONAL methods of vibration suppression make the structure as rigid as possible, approximate the residual damping as viscous, and then design an optimal controller with respect to a predetermined cost function.[1] Although the controller effectively stabilizes modes at specific frequencies, problems arise when the structure has modes at closely spaced frequencies. This is typically the case with large space structures, which often have closely spaced, low resonant frequencies. To stabilize the associated modes, passive dampers are added.[2] But this changes the whole system: How "optimum" is the controller now? Is control effort still optimized? What is the "best" size for the passive dampers? If one does not design passive and active control elements simultaneously, higher weight costs are likely, which is important for space structures.

To overcome these problems, Mar encouraged viewing damping as a creative force in design.[3] Along these lines, several recent research efforts have been aimed at designing a damped structure, then designing a controller for it.[4-6] Simonian et al.[7] point out that incorporating passive damping in an active control design must be done so that the deficiencies of one technology are compensated by the strengths of the other. They propose an iterative scheme that uses modal strain energy analysis to determine the modal damping. A hybrid cost function is used to determine the tradeoffs. Fowler et al. implemented a variation of this method in a computer program.[8] Grandhi assumes proportional damping and uses a computer program to minimize weight.[9] Onoda and Watanabe use a direct numerical optimization approach for design of an optimal controller incorporated into a structure/controller simultaneous optimization scheme but do not specifically address damping issues.[10]

None of the cited methods attempt closed-form simultaneous optimization of passive viscous damping and active control, although Gibson[11] does point out the need for it. In an effort to improve design methods, a new method of simultaneous optimization is proposed.

Optimizing Viscous Damping and Control Simultaneously

To determine an "optimum" for damping and control, one must first select a suitable performance index. A common one used in controls is the linear quadratic regulator (LQR) cost functional.[12]

To optimize damping and control simultaneously, consider a variation of the LQR cost function. Let v be a vector of passive control forces in the system, and S its positive definite weighting matrix. Then

$$J = \int_{t_0}^{t_f} \frac{1}{2}(x^T Q x + u^T R u + v^T S v)\, \mathrm{d}t \tag{1}$$

and

$$\dot{x} = Ax + Bu + B_v v \tag{2}$$

The passive damping and associated stiffness coefficients have been taken out of the term Ax, in which they normally reside, and placed in a separate term $B_v v$. The passive forces are directly analogous to the active control forces and are weighted in the same manner.

We have specific reasons for weighting the passive forces like the active control forces. When implementing a damping design using viscous fluid dampers or viscoelastic solids, one must take into account the temperature sensitivity of the damping medium. Modest changes in temperature due to absorbed mechanical energy can dramatically alter the damping properties of both fluids and solids. Hence we are motivated to limit in some fashion the mechanical energy absorbed by any given damper as well as limit the peak value of its damping force. The quadratic damping term appearing in the performance index, Eq. (1), serves this end rather nicely. One should note that this quadratic damping term is reminiscent of the Rayleigh dissipation function used in conjunction with energy methods in classical dynamics.

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Assume v is of the form

$$v = C\Phi x \tag{3}$$

where C contains the desired passive damping coefficients and perhaps some stiffness coefficients. The matrix Φ is chosen such that C is diagonal. In some cases, the matrix Φ can be chosen to be invertible, as is the case for one of the example problems later in the paper.

Using the method of Lagrange multipliers to append the constraints to the performance index gives

$$J = \int_{t_0}^{t_f} [\tfrac{1}{2}(x^TQx + u^TRu + v^TSv) - \lambda_1^T(\dot{x} - Ax - Bu - B_v v) - \lambda_2^T(v - C\Phi x)]\, dt \tag{4}$$

Now x, u, and v are taken to be independent, which implies C is independent of these quantities also. By definition, the vectors λ_1 and λ_2 are independent of each other and of x, u, v, and C. Hence, to minimize J, take its variation and set it equal to zero. The variation of J is

$$\begin{aligned}\delta J = \int_{t_0}^{t_f} & [(x^TQ\delta x + u^TR\delta u + v^TS\delta v) \\ & - \delta\lambda_1^T(\dot{x} - Ax - Bu - B_v v) \\ & - \lambda_1^T(\delta\dot{x} - A\,\delta x - B\delta u - B_v\delta v) - \delta\lambda_2^T(v - C\Phi x) \\ & - \lambda_2^T(\delta v - \delta C\Phi x - C\Phi\delta x)]\, dt\end{aligned}$$

Since initial and final conditions are specified,

$$\int_{t_0}^{t_f} \lambda_1^T\delta\dot{x}\, dt = \lambda_1^T\delta x \,|_{t_0}^{t_f} - \int_{t_0}^{t_f} \dot{\lambda}_1^T\delta x\, dt = -\int_{t_0}^{t_f} \dot{\lambda}_1^T\delta x\, dt \tag{5}$$

Then δJ becomes

$$\begin{aligned}\delta J = \int_{t_0}^{t_f} & [(x^TQ + \dot{\lambda}_1^T + \lambda_1^TA + \lambda_2^TC\Phi)\delta x + (u^TR + \lambda_1^TB)\delta u \\ & + (v^TS + \lambda_1^TB_v - \lambda_2^T)\delta v - \delta\lambda_1^T(\dot{x} - Ax - Bu - B_v v) \\ & - \delta\lambda_2^T(v - C\Phi x) + \lambda_2^T(\delta C)\Phi x]\, dt\end{aligned} \tag{6}$$

The independence of the six unknowns leads to the six equations

$$\dot{x} = Ax + Bu + B_v v \tag{7}$$

$$v = C\Phi x \tag{8}$$

$$u = -R^{-1}B^T\lambda_1 \tag{9}$$

$$0 = \dot{\lambda}_1 + A^T\lambda_1 + Qx + \Phi^TC\lambda_2 \tag{10}$$

$$\lambda_2 = B_v^T\lambda_1 + Sv \tag{11}$$

$$\lambda_2 = 0 \tag{12}$$

Equations (11) and (12) give

$$v = -S^{-1}B_v^T\lambda_1 \tag{13}$$

Substituting Eqs. (9), (12), and (13) into Eqs. (7) and (10) yields

$$\dot{x} = Ax - BR^{-1}B^T\lambda_1 - B_vS^{-1}B_v^T\lambda_1 \tag{14}$$

$$\dot{\lambda}_1 = -Qx - A^T\lambda_1 \tag{15}$$

These two equations can be written as one matrix equation

$$\begin{Bmatrix}\dot{x}\\ \dot{\lambda}_1\end{Bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T - B_vS^{-1}B_v^T \\ -Q & -A^T\end{bmatrix}\begin{Bmatrix} x \\ \lambda_1\end{Bmatrix} \tag{16}$$

This equation can be solved for x and λ_1 for given $x(t_0)$ and $x(t_f)$. Now Eqs. (8) and (13) give

$$C\Phi x = -S^{-1}B_v^T\lambda_1 \tag{17}$$

Since C is diagonal, the ith diagonal element of C (c_i) can be written as

$$c_i = \frac{[-S^{-1}B_v^T\lambda_1]_i}{[\Phi x]_i} \tag{18}$$

where the subscript i denotes the ith element of the indicated vector. Not only does the preceding relation indicate that C is time varying, it also indicates that if any element of Φx is zero at any time, the corresponding c_i is infinity. Since it is desirable that the passive damping coefficients be constant (and finite), we will attempt to find a solution of the form of the standard LQR solution.

Minimizing the Cost Functional Error

For now, assume λ_1 is of the form $\lambda_1 = Px$. (We will see later that this yields an optimal solution that contradicts the requirement that C is diagonal.) This is the form of the standard LQR solution. Then, with $\lambda_2 = 0$, Eq. (10) becomes

$$0 = \dot{P}x + P\dot{x} + A^TPx + Qx \tag{19}$$

whereas Eq. (11) becomes

$$Sv = -B_v^TPx \tag{20}$$

or

$$v = -S^{-1}B_v^TPx \tag{21}$$

Equivalently,

$$C\Phi x = -S^{-1}B_v^TPx \tag{22}$$

or

$$C\Phi = -S^{-1}B_v^TP \tag{23}$$

Substituting for $\dot{x}$, u, and v in Eq. (19) gives

$$\dot{P}x + P(Ax + Bu + B_v v) + A^TPx + Qx = 0$$

$$\dot{P}x + PAx + P[B(-R^{-1}B^TPx) + B_v(-S^{-1}B_v^TPx)] + A^TPx + Qx = 0$$

$$[\dot{P} + PA + A^TP - P(BR^{-1}B^T + B_vS^{-1}B_v^T)P + Q]x = 0$$

which leads to the Riccati equation

$$\dot{P} + PA + A^TP - P(BR^{-1}B^T + B_vS^{-1}B_v^T)P + Q = 0$$

Restricting attention to the steady-state solution, $\dot{P} = 0$, which is equivalent to letting $t_f \to \infty$ (Ref. 12), leads to the algebraic Riccati equation.

$$PA + A^TP - P(BR^{-1}B^T + B_vS^{-1}B_v^T)P + Q = 0 \tag{24}$$

If Φ is invertible, from Eq. (23), C is determined by the relation

$$C = -S^{-1}B_v^TP\Phi^{-1} \tag{25}$$

where P satisfies the algebraic Riccati equation. This solution for C is in general not diagonal. As originally posed, however, C must be a positive semidefinite diagonal matrix. This requirement was not specifically addressed in the preceding derivation. This constraint is normally not encountered in an LQR problem. Consequently, there is no standard technique for incorporating it. In general C, as computed from Eq. (25), is fully populated, but initial numerical solutions indicate that the diagonal terms of interest are significantly larger than the off-diagonal terms. The assumption that the solution has the form of the standard regulator (i.e., $\lambda_1 = Px$) is what led us to a nondiagonal C. Unlike the standard LQR problem, the solution to the optimization problem posed by Eq. (1) and C constrained positive definite diagonal is not given by a Riccati equation, even when $t_f \to \infty$. We will demonstrate later that the solution to the present problem will, however, require an iteration involving a Riccati equation similar to Eq. (24).

If the assumption that $\lambda_1 = Px$ is not made, a diagonal C can be obtained, but it is time dependent as shown in Eq. (18). In addition, the feedback gains for the active control forces u are also time dependent. Since it is desirable to maintain constant feedback gains, as well as constant damping coefficients, suboptimal solutions are sought with constant diagonal C and constant active feedback gains that yield the lowest cost functional.

A good estimate of C that is diagonal would be the diagonal matrix that is the best least squares fit to Eq. (23). Equivalently, minimize the Frobenius norm

$$\|C\Phi + S^{-1}B_v^T P\|_F \tag{26}$$

where P is the solution of Eq. (24). (The Frobenius norm of a matrix A is defined as the square root of the sum of the squares of the elements of the matrix $\|A\|_F = [\Sigma\Sigma A_{ij}^2]^{1/2}$.) This is a linear minimization problem and has a closed-form solution (see Appendix for derivation). If

$$W = -S^{-1}B_v^T P \tag{27}$$

the elements of C are

$$c_{ii} = \frac{\sum_{j=1}^{n} \Phi_{ij} W_{ij}}{\sum_{j=1}^{n} \Phi_{ij}^2}; \qquad c_{ij} = 0 \text{ for } i \neq j \tag{28}$$

If m is the number of damper coefficients, and n is the length of y, Φ and W are $m \times n$ matrices. This solution also has the added attraction of guaranteed positive values of c_{ii} (see Appendix).

Let us step back for a moment and examine the minimum solution of Eq. (1) when C is chosen a priori as opposed to simultaneously. The cost functional becomes

$$J = \int_{t_0}^{t_f} \frac{1}{2}[x^T(Q + \Phi^T CSC\Phi)x + u^T Ru]\, dt \tag{29}$$

subject to

$$\dot{x} = (A + B_v C\Phi)x + Bu \tag{30}$$

We see with C fixed in advance, the minimization is a standard LQR problem with modified Q and A matrices. The minimum J is achieved by

$$u = -R^{-1}B^T Px \tag{31}$$

where P (in the limit as $t_f \to \infty$) is now the solution of

$$P(A + B_v C\Phi) + (A + B_v C\Phi)^T P - PBR^{-1}B^T P + Q + \Phi^T CSC\Phi = 0 \tag{32}$$

It can be shown[12] that as $t_f \to \infty$, the value of the cost functional is

$$J = \lim_{t_f \to \infty} J(t_f) = \tfrac{1}{2} x_0^T P x_0 \tag{33}$$

Therefore, if x_0 is known, we can iterate on a diagonal C [using Eq. (32) to determine P during each iteration] to minimize J in Eq. (33). However, optimizing for a single fixed x_0 may result in poor off design performance. In contrast, the standard LQR problem does not require this iteration because P is independent of x_0, and hence results in inherently robust performance. To improve the performance robustness, we will consider solution techniques independent of x_0.

To get a measure of the difference between the value of J that is achievable with an unconstrained C and the J achieved with constrained C, note that

$$\Delta J = x_0^T \hat{P} x_0 - x_0^T P x_0 = x_0^T(\hat{P} - P)x_0 = x_0^T \Delta P x_0 \tag{34}$$

where P represents the unconstrained optimal solution and $\hat{P}$ a constrained solution. Now P satisfies

$$PA + A^T P - P(BR^{-1}B^T + B_v S^{-1}B_v^T)P + Q = 0 \tag{35}$$

whereas $\hat{P}$ satisfies

$$\hat{P}(A + B_v C\Phi) + (A + B_v C\Phi)^T \hat{P} - \hat{P}BR^{-1}B^T\hat{P} + Q + \Phi^T CSC\Phi = 0 \tag{36}$$

Subtracting Eq. (35) from Eq. (36),

$$\begin{aligned}\Delta PA &+ A^T\Delta P + \Delta PB_vC\Phi + PB_vC\Phi + (B_vC\Phi + PB_vC\Phi)^T\Delta P \\ &+ (B_vC\Phi + PB_vC\Phi)^T P - \Delta P(BR^{-1}B^T P) - PBR^{-1}B^T P \\ &- \Delta P(BR^{-1}B^T)\Delta P + \Phi^T CSC\Phi + PB_vS^{-1}B_v^T P = 0\end{aligned} \tag{37}$$

This can be written in the form of a Riccati equation

$$\begin{aligned}\Delta P(A - BR^{-1}B^T P + B_v C\Phi) &+ (A - BR^{-1}B^T P \\ &+ B_v C\Phi)^T \Delta P - \Delta P(BR^{-1}B^T)\Delta P + E = 0\end{aligned} \tag{38}$$

where $E = \Phi^T CSC\Phi + \Phi^T CB_v^T P + PB_v C\Phi + PB_v S^{-1}B_v^T P$. This Riccati equation determines ΔP, which in turn determines ΔJ. Note that $(A - BR^{-1}B^T P + B_v C\Phi)$ is stable. Therefore, if $E = 0$, then $\Delta P = 0$ is the only real symmetric solution to Eq. (38).[13] Therefore, driving E as small as possible yields a small ΔP and thus drives $\hat{P}$ to P. This is highly desirable because we know that P (the unconstrained standard LQR solution) is highly robust with respect to initial conditions.

Note that E can be written as

$$\begin{aligned}E &= \Phi^T C(SC\Phi + B_v^T P) + PB_v S^{-1}(SC\Phi + B_v^T P) \\ &= (\Phi^T C + PB_v S^{-1})(SC\Phi + B_v^T P) \\ &= (\Phi^T CS + PB_v)S^{-1}(SC\Phi + B_v^T P) \\ &= [S^{-1/2}(SC\Phi + B_v^T P)]^T[S^{-1/2}(SC\Phi + B_v^T P)] \\ &= H^T H\end{aligned} \tag{39}$$

where $H = S^{-1/2}(SC\Phi + B_v^T P)$. Therefore, choosing C to make $SC\Phi$ as close to $-B_v^T P$ as possible will make E small. Note that $\|H^T H\|_2 = \|H\|_2^2$. Thus choosing a C that minimizes the two norm of H minimizes the two norm of E and drives $\Delta J \to 0$. This minimization can be accomplished using the MATLAB routine FMINS. FMINS uses the Nelder-Mead simplex search algorithm.[14]

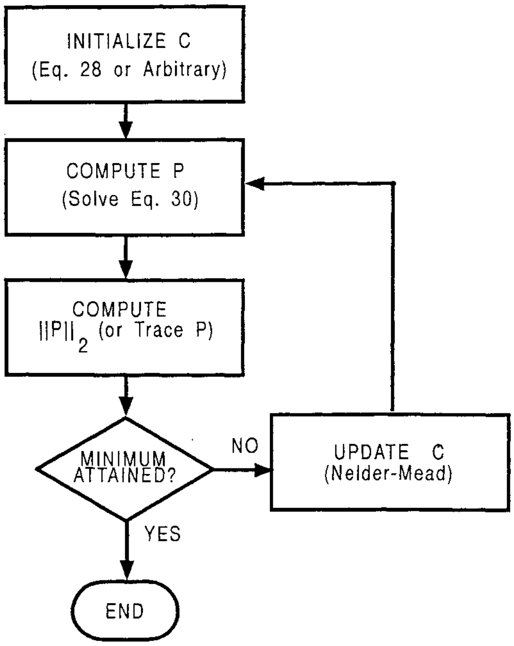

Fig. 1 Iteration procedure.

Notice how similar H is to the error term in Eq. (26). In fact, minimizing the Frobenius norm of Eq. (26) gives the same result as minimizing $\|H\|_F$. The advantage of this method over minimizing $\|H\|_2$ is that it is guaranteed to return positive damping coefficients and can be solved in closed form (see Appendix).

Alternative J Optimization

Note that the combination of C and P that minimizes J [Eq. (33)] is independent of the magnitude of the vector x_0. Therefore, comparing solutions for $\|x_0\| = 1$ reflects the performance over the entire space.

One alternative approach is to minimize the maximum value of J for all $\|x_0\| = 1$. This approach requires determining the two norm of P, since

$$J = ½ \mid x_0^T P x_0 \mid \leq ½ \|x_0\|^2 \|P\|_2 = ½ \|P\|_2 \tag{40}$$

In this method, iteration with respect to C is carried out until $\|P\|_2$ is minimized. The flowchart in Fig. 1 outlines the iteration procedure. This approach might be very conservative in general, since the x_0 that maximizes P may not be encountered very often.

Another alternative approach is to minimize the average value of J over the unit ball, $\|x_0\| = 1$. Let $x_0 = [x_1\, x_2 \cdots x_n]^T$. Since P is symmetric,

$$J = \frac{1}{2} x_0^T P x_0 = \frac{1}{2} \sum_{i=1}^{n} P_{ii} x_i^2 + \sum_{i,j=1}^{n} P_{ij} x_i x_j \tag{41}$$

and

$$\bar{J} = \frac{\int_{\|x_0\|=1} \left(\frac{1}{2} \sum_{i=1}^{n} P_{ii} x_i^2 + \sum_{i,j=1}^{n} P_{ij} x_i x_j \right) \mathrm{d}A}{\int_{\|x_0\|=1} \mathrm{d}A}$$

$$= \frac{\left[\frac{1}{2} \sum_{i=1}^{n} P_{ii} \left(\int_{\|x_0\|=1} x_i^2 \, \mathrm{d}A \right) + \sum_{ij=1}^{n} P_{ij} \left(\int_{\|x_0\|=1} x_i x_j \, \mathrm{d}A \right) \right]}{\int_{\|x_0\|=1} \mathrm{d}A} \tag{42}$$

The divergence formula is

$$\int_T \operatorname{div} u \, \mathrm{d}V = \int_s n \cdot u \, \mathrm{d}A \tag{43}$$

Let e_i represent the unit vector in the direction of the x_i coordinate. Then the unit vector normal to the surface of the unit ball is

$$n = \sum_{i=1}^{n} x_i e_i$$

In the first integral in Eq. (42), $n \cdot u = x_i^2$. This implies that $u = x_i e_i$. Therefore,

$$\operatorname{div} u = \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k} = \frac{\partial x_i}{\partial x_i} = 1 \tag{44}$$

Therefore, evaluation of the first integral in Eq. (42) is

$$\int_{\|x_0\|=1} x_i^2 \, \mathrm{d}A = \int_{\|x_0\|=1} \mathrm{d}V = V \tag{45}$$

the volume of the unit ball.

In the second integral in Eq. (42), $n \cdot u = x_i x_j$. This requires $u = ½(x_i e_j + x_j e_i)$. Then,

$$\operatorname{div} u = \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k} = \frac{\partial(½ x_j)}{\partial x_i} + \frac{\partial(½ x_i)}{\partial x_j} = 0 \tag{46}$$

Hence,

$$\int_{\|x_0\|=1} x_i x_j \, \mathrm{d}A = 0 \tag{47}$$

The last integral in Eq. (42) is just the surface area of the unit ball, but it can be expressed in terms of the volume. The quantity $n \cdot u = 1$. This requires that $u = n$. Thus,

$$\operatorname{div} u = \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k} = \sum_{k=1}^{n} \frac{\partial x_k}{\partial x_k} = n \tag{48}$$

This gives

$$\int_{\|x_0\|=1} \mathrm{d}A = \int_{\|x_0\|=1} n \, \mathrm{d}V = nV \tag{49}$$

Combining Eqs. (42), (45), (47), and (49),

$$\bar{J} = \frac{1}{2n} \operatorname{tr} P \tag{50}$$

Hence, minimizing the trace of P minimizes the average value of J over the unit ball $\|x_0\| = 1$, regardless of system order. This is accomplished by iteration with respect to C (see Fig. 1). A similar derivation of minimum average control energy is given by Kalman et al.[15]

We have described four solution techniques that are independent of initial conditions. In summary, the four solution methods are:

1) min $\|H\|_F$: Closed-form solution based on finding a diagonal C that minimizes the error in the cost with respect to the Frobenius norm; see Eqs. (34–39).

2) min $\|H\|_2$: Iterative solution technique based on finding a diagonal C that minimizes the error in the cost with respect to the two norm; see Eqs. (34–39).

3) min $\|P\|_2$: Iterative solution technique based on finding a diagonal C that minimizes the maximum value of the cost; see Eq. (40).

4) min (trace P): Iterative solution technique based on finding a diagonal C that minimizes the average value of the cost; see Eqs. (41–50).

Table 1 Damping coefficients for 2-DOF system

Case	c
x_0 fixed	0.79
min $\|P\|_2$	0.40
min tr P	0.46
min $\|H\|_F$	0.71
min $\|H\|_2$	0.71

Example Problem 1

Consider a spring-mass-damper system whose equation of motion is given by the scalar equation

$$\ddot{z} + c\dot{z} + (0.5 + \Delta k)z = -u \tag{51}$$

where Δk and c are the passive control design parameters. Letting $x_1 = z$, $x_2 = \dot{z}$, then

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -0.5 & 0 \end{bmatrix} x + \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} u + \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \Delta k & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \tag{52}$$

$$\dot{x} = Ax + Bu + B_v C \Phi x \tag{53}$$

Let the weighting matrices be

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad R = 1, \qquad S = \begin{bmatrix} 1000 & 0 \\ 0 & 1 \end{bmatrix} \tag{54}$$

The weight on Δk is chosen large to assure negligible extra passive stiffness. This optimal unconstrained C [i.e., the solution of Eq. (23)] for this problem is

$$C = \begin{bmatrix} 0 & 0 \\ 0.50 & 0.71 \end{bmatrix} \tag{55}$$

If we pick $x_0 = [0\ 1]^T$ and iterate on diagonal C to minimize Eq. (33) subject to Eq. (32), we get

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0.79 \end{bmatrix} \tag{56}$$

$$J = 0.7860 \tag{57}$$

where the J for the optimal unconstrained C and $x_0 = [0\ 1]^T$ would be 0.7069.

For the two-degree-of-freedom case, $\|x_0\| = 1$ can be represented by a single parameter θ where $x_0 = [\cos\theta\ \sin\theta]^T$. The off design results for the optimal unconstrained C is shown as a solid line with circles in Fig. 2. The results for diagonal C minimizing J for $x_0 = [0\ 1]^T$ are shown also. Also included for comparison is the result for purely active damping (i.e., $C = 0$). (The design results are the results at $\theta = \pi/2$.)

Notice that the system optimized at $x_0 = [0\ 1]^T$ does not always outperform the pure active damping case for off design x_0. Clearly, optimizing for a fixed x_0 will not always be robust (i.e., insensitive to changes in x_0). Let's now examine the alternative suboptimal solutions that are independent of x_0. Though not strictly optimal, these solutions are much more robust to off design x_0.

For the two-degree-of-freedom problem, minimizing $\|P\|_2$ resulted in a diagonal C of

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0.40 \end{bmatrix} \tag{58}$$

The result is plotted in Fig. 2. The graph shows this solution to be much more robust than optimizing for a given x_0. All other solutions will have some values of J above the maximum of this one.

Minimizing the trace of P for Example 1 gave a diagonal C of

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0.46 \end{bmatrix} \tag{59}$$

These results are also shown in Fig. 2. Although this graph does have points above the one for min $\|P\|_2$, the average value of J is lower for this solution than for any other. The min $\|H\|_2$ and the min $\|H\|_F$ solutions are not shown on Fig. 1. They are very close to the $c = 0.79$ solution.

Note that both the solution minimizing the average value of the cost functional and the one minimizing the maximum value of the cost functional outperform the solution using active control only. Hence the simultaneous design of active vibration control and passive damping is superior to active control alone.

Table 1 summarizes various results for the two-degree-of-freedom system. For this example, the results for the min $\|H\|_2$ and min $\|H\|_F$ cases are the same to two decimal places, but in general the results will be different. Although the damping coefficients for the min $\|H\|_2$ and min $\|H\|_F$ cases are closer to the fixed x_0 case, they nonetheless serve as decent first guesses for the min $\|P\|_2$ and the min (tr P) cases. The importance of this in higher order systems will become evident in Example 2.

Example Problem 2

Consider the two-dimensional aluminum truss in Fig. 3. The finite element model equations of motion can be written in the state-space form

$$\dot{x} = Ax + Bu + B_v C \Phi x \tag{60}$$

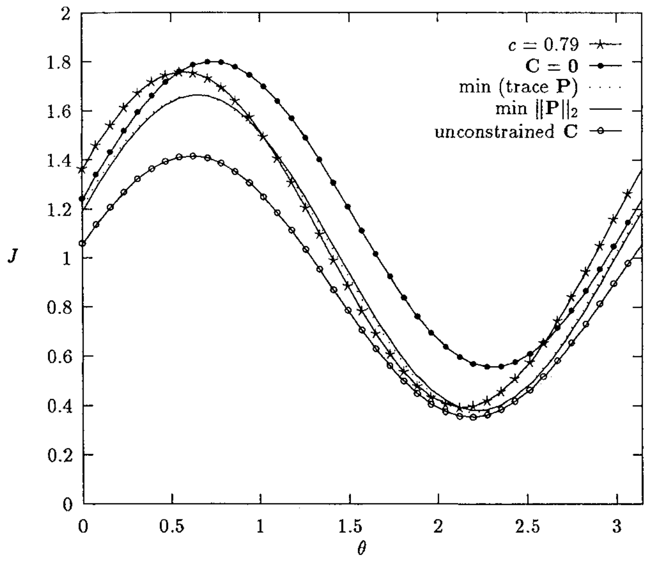

Fig. 2 Performance index vs initial conditions.

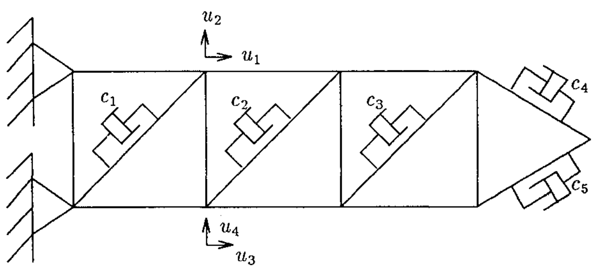

Fig. 3 Example problem 2 truss structure.

Table 2 Damping coefficients and CPU times for 14-DOF truss

min $\|H\|_F$	min $\|H\|_2$	min$_0$ $\|H\|_2$	min tr P	min$_0$ tr P	min $\|P\|_2$	min$_0$ $\|P\|_2$
11.7599	17.9135	17.9384	10.7839	10.7837	18.7650	18.7657
7.1387	37.4876	−1.9959	6.6476	6.6476	10.0413	10.0411
7.8167	9.1892	9.1836	7.6351	7.6351	7.0617	7.0617
4.8881	6.7594	6.7571	6.4359	6.4358	7.4100	7.4101
4.6872	5.9470	5.9454	6.3189	6.3185	9.0092	9.0092
2 s	30 s	27 s	17 min	33 min	23 min	102 min

Table 3 Damping factors and natural frequencies for 14-DOF truss

Undamped poles, 10^4	min $\|H\|_F$ ζ	min $\|H\|_F$ ω_n, 10^4 rad/s	min tr P ζ	min tr P ω_n, 10^4 rad/s	min $\|P\|_2$ ζ	min $\|P\|_2$ ω_n, 10^4 rad/s
1.8725i	0.1944	1.8477	0.2584	1.8285	0.3497	1.8008
1.5261i	0.1313	1.4845	0.1347	1.4935	0.1812	1.4903
1.4637i	0.0811	1.4609	0.0884	1.4482	0.0961	1.4316
1.2525i	0.1509	1.2276	0.1655	1.2247	0.2108	1.2076
1.2246i	0.1214	1.2238	0.1185	1.2358	0.1764	1.1995
1.1299i	0.0206	1.1345	0.0229	1.1370	0.0929	1.1738
1.1227i	0.0652	1.1388	0.0703	1.1378	0.0256	1.1401
0.8670i	0.0564	0.8788	0.0625	0.8836	0.0745	0.8991
0.7845i	0.0472	0.7902	0.0488	0.7912	0.0577	0.7944
0.6004i	0.0993	0.6063	0.0959	0.6062	0.1038	0.6025
0.4408i	0.1204	0.4465	0.1144	0.4458	0.1684	0.4521
0.1750i	0.1083	0.1768	0.1073	0.1767	0.1162	0.1780
0.2687i	0.0496	0.2689	0.0487	0.2689	0.0463	0.2407
0.2400i	0.0446	0.2404	0.0446	0.2404	0.0594	0.2691

The state-space vector x is 28×1. There are four control inputs and five damping coefficients, therefore u is 4×1 and C is 5×5. Therefore, A is 28×28, B is 28×4, B_v is 28×5, and Φ is 5×28. The parameters of the system are

$$EA = 1.822 \times 10^7 \text{ N} \quad (4.095 \times 10^6 \text{ lb})$$

$$\rho A = 0.6964 \text{ N-s}^2/\text{m}^2 \quad (1.010 \times 10^{-4} \text{ lb-s}^2/\text{in.}^4)$$

The lengths of the horizontal and vertical elements, and of the two elements with dampers c_4 and c_5 are 0.6096 m (2.0 ft). Thus, the diagonal elements with dampers c_1, c_2, and c_3 are 0.8621 m (2.828 ft) in length. The state weighting matrix Q was chosen such that x^TQx equalled the total mechanical energy in the system. Thus Q was formed using the mass and stiffness matrices, M and K, whereas the passive damping weighting matrix S was chosen to be $10^{-4} \times$ identity, and the control weighting matrix R was chosen to be $10^{-3} \times$ identity

$$Q = \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix}, \qquad R = 10^{-3}I, \qquad S = 10^{-4}I \tag{61}$$

The resulting damping coefficients for the different solution techniques are given in Table 2. At the bottom of Table 2 are the approximate computation times on a VAX 6420. The solution techniques were run with simple MATLAB routines, using default tolerances in the minimization routines. The minimization algorithm utilized is the Nelder-Mead simplex method.[14] The min $\|H\|_F$ solution was used as an initial guess for the other solutions. (The CPU time for the min $\|H\|_F$ solution is included for completeness, but since it is a closed-form solution, it can be computed before numerical minimizations are performed.)

Shown in Table 3 are the open-loop poles of the system (no active or passive control) and the closed-loop damping factors and natural frequencies using the min (tr P) and min $\|P\|_2$ solution methods. The zero subscript on min indicates that an initial guess of $C = 0$ was used. The other cases used the min $\|H\|_F$ solution as an initial guess. This significantly decreased the computation time, except in the min $\|H\|_2$ case. Since the min $\|H\|_2$ case gave different answers for the different initial guesses, the relative CPU times have little meaning. In fact, the negative damping coefficient returned by the zero initial guess indicates this is not a realistic solution, and so it must be discarded. Also, the significant difference in c_2 from the other solutions for the Frobenius norm initial guess implies that this solution may not be very realistic either. Recall that the min $\|H\|_F$ solution will always return positive damping coefficients and is the minimum least squares fit to the unconstrained solution (see Appendix). If one wishes to guarantee positive solutions for the iterative techniques, the constraint that the damping coefficients must be positive can be incorporated into their program codes.

The results in Table 2 clearly demonstrate that if one chooses not spend the additional computer time to optimize the dampers that the inexpensive min $\|H\|_F$ solution is a reasonably close estimate. For very large systems this solution may be the only affordable alternative. Figure 4 shows the gain and phase plots of the structure's tip response in the vertical direction to a vertical input near the base of the structure, input u_4. The three solutions shown [min $\|H\|_F$, $\|P\|_2$, and min, (tr P)] give almost identical responses. This supports the conclusion that the min $\|H\|_F$ solution is a reasonable estimate. The closed-loop damping factors for the min $\|H\|_F$, $\|P\|_2$, and min (tr P) solutions are shown in Table 3.

Since we are unable to show a two-dimensional plot of J over the unit ball as we did in Example 1, we need another method of comparing the solutions. Recall from Eq. (34) that the difference in J between two different solutions can be given by

$$\Delta J = x_0^T(P_1 - P_2)x_0 \tag{62}$$

where P_1 and P_2 are two different solutions. If the matrix $(P_1 - P_2)$ is positive definite, then $\Delta J > 0$ for all x_0. Hence, P_2 would give a lower value of J than P_1 for all x_0. As shown in Table 4, the min $\|H\|_F$ solution (P_F), the min (tr P) solution (P_{tr}), and the min $\|P\|_2$ solution (P_2) give lower values of J for all x_0 than the solution using only active damping (P_{c0}). Only 14 eigenvalues are shown as the second 14 were essentially zero. Table 4 also shows that P_{tr} gives a lower value of J than P_2 in seven directions, and a higher value of J in the other seven directions. Since most of the positive eigenvalues of ($P_2 - P_{tr}$) are greater in magnitude than the negative eigenvalues, it can be argued that P_{tr} is a better overall solution than P_2.

The cost functional of Eq. (1) treats the passive damping forces as if they were similar to the active damping forces. It can be argued that passive damping is an initial one-time cost item and ought to be weighed as such. This leads to a cost functional of the form

$$J = \frac{1}{2}c^TSc + \int_{t_0}^{\infty} \frac{1}{2}(x^TQx + u^TRu)\,dt \tag{63}$$

subject to

$$\dot{x} = (A + B_vC\Phi)x + Bu \tag{64}$$

where c is a vector of the passive damping parameters and C is a diagonal matrix with the elements of c on its main diagonal. Regardless of the values of c, the minimum of the integral term is given by $\frac{1}{2}x_0^TPx_0$ where P satisfies

$$P(A + B_vC\Phi) + (A + B_vC\Phi)^TP - PBR^{-1}B^TP + Q = 0 \tag{65}$$

To minimize the maximum value of J we need to minimize

$$J = \tfrac{1}{2}\,|\,x_0Px_0 + c^TSc\,| \le \tfrac{1}{2}(\|x_0\|^2\|P\|_2 + c^TSc)$$

$$= \tfrac{1}{2}(\|P\|_2 + c^TSc) \tag{66}$$

for $\|x_0\| = 1$, where P is found by solving Eq. (65).

To minimize the average value of J we need to minimize

$$\bar{J} = \frac{\frac{1}{2}\int_{\|x_0\|=1} x_0^TPx_0\,dA}{\int_{\|x_0\|=1} dA} + \frac{\frac{1}{2}\int_{\|x_0\|=1} c^TSc\,dA}{\int_{\|x_0\|=1} dA} \tag{67}$$

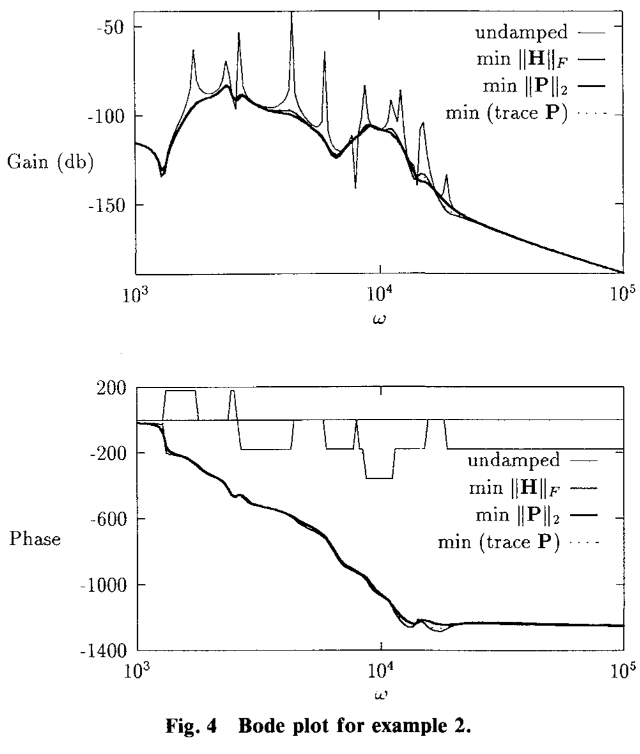

Fig. 4 Bode plot for example 2.

Table 4 Eigenvalues of Delta P for 14-DOF truss

$P_{c0} - P_F$	$P_{c0} - P_{tr}$	$P_{c0} - P_2$	$P_2 - P_{tr}$
8064.7	8025.3	7950.6	121.8
2920.9	2921.3	2856.7	87.5
2797.3	2802.1	2785.9	−47.1
2569.2	2579.1	1565.1	59.2
1982.6	2025.4	2025.3	32.7
1165.3	1173.5	1101.3	−26.5
729.2	736.3	727.5	16.9
609.5	618.5	599.2	−21.3
526.2	541.2	550.3	9.6
449.7	459.2	462.0	7.6
140.6	139.1	145.6	−8.0
83.3	80.1	102.4	−5.7
19.3	18.7	23.3	−4.5
14.3	15.3	19.8	−1.9

$$\bar{J} = \frac{1}{2n}\,\text{tr}\,P + \frac{1}{2}c^TSc \tag{68}$$

where P is again given by Eq. (65).

Conclusions

Simultaneous design to determine optimal blending of passive damping and active vibration control has been considered. Four techniques based on modified versions of the standard linear quadratic regulator cost functional of optimal control theory were developed. Two of the techniques (one of which is closed form) attempt to minimize the error in the cost due to using viscous dampers rather than a true active control force. The closed-form solution can be used as a starting estimate for the damping parameters in two additional iterative techniques, one that minimizes the average energy over the range of possible initial conditions and one that minimizes the maximum possible energy.

Appendix

In this Appendix we derive the solution to the diagonal C that minimizes the Frobenius norm of Eq. (25). Let X be the space of vectors with four elements and let Y be the space of 4×4 matrices. Define an operator $A: X \to Y$ by

$$y = A(x) = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{bmatrix} \tag{A1}$$

where $x \in X$. It is easily verified that $A(x + \alpha w) = A(x_1) + \alpha A(w)$ for any scalar α and any $x, w \in X$, so A is linear.

Define a functional on Y by

$$(y, z) = \sum_{i,j=1}^{4} y_{ij}z_{ij} \;\forall\, y, z \in Y \tag{A2}$$

Since this functional satisfies the four requirements of an inner product,

$$(y, z) = (z, y)$$

$$(y_1 + y_2, z) = (y_1, z) + (y_2, z)$$

$$(\lambda y, z) = \lambda(y, z)$$

$$(y, y) \ge 0, \qquad (y, y) = 0 \text{ iff } y = 0$$

define the functional in Eq. (A2) to be the inner product on Y. Then the norm on Y is given by

$$\|y\| = (y, y)^{1/2} = \left[\sum_{i,j=1}^{4} y_{ij}^2\right]^{1/2} \tag{A3}$$

Notice that this is the Frobenius norm for 4×4 matrices.

The conjugate operator $A^*: Y \rightarrow X$ is found from the relationship

$$(x, A^*y)_X = (Ax, y)_Y \tag{A4}$$

The subscripts X and Y denote the spaces in which the inner product is taken. Since

$$(Ax, y)_Y = \sum_{i,j=1}^{4} (Ax)_{ij} y_{ij} = \sum_{i=1}^{4} x_i y_{ii} = (x, A^*y)_X \tag{A5}$$

the conjugate operator is given by

$$A^*y = \begin{Bmatrix} y_{11} \\ y_{22} \\ y_{33} \\ y_{44} \end{Bmatrix} \tag{A6}$$

The solution that yields the smallest error in the Frobenius norm is given by $(A^*A)^{-1}A^*y$ if $(A^*A)^{-1}$ exists.[16] Find A^*A

$$A^*Ax = A^* \left(\begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{bmatrix} \right) = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = x \tag{A7}$$

So A^*A is the identity operator and its inverse is also the identity operator. Hence the solution that minimizes the Frobenius norm is

$$x = A^*y \tag{A8}$$

In other words, x is the diagonal of y.

Using the same procedure just outlined, it can be shown that the diagonal C that minimizes the Frobenius norm $\|C\Phi - y\|_F$ is given by Eq. (28).

In the case where C contains only viscous damping coefficients (no spring stiffnesses),

$$B_v = \begin{bmatrix} 0 & 0 \\ 0 & -M^{-1} \end{bmatrix} \Phi^T \tag{A9}$$

Let

$$\begin{bmatrix} 0 & 0 \\ 0 & M^{-1} \end{bmatrix} P = \hat{P} \tag{A10}$$

Since M is positive definite and P is positive semidefinite, $\hat{P}$ is positive semidefinite. Assuming $\hat{P}$ has n independent eigenvectors, $\hat{P} = E\Lambda E^T$ where E is the orthonormal matrix of eigenvectors and Λ is the diagonal matrix of eigenvalues. Then $y = -S^{-1}\Phi E\Lambda E^T$. Defining S_i as the ith element of the diagonal of S, λ_j as the jth eigenvalue, e_j as the jth eigenvector, and ϕ_i as the ith row of Φ, the ith element of c is given by

$$c_i = \frac{1}{S_i \sum_{j=1}^{n} \phi_{ij}^2} \left(\sum_{j=1}^{n} \lambda_j (\phi_i e_j)^2 \right) \tag{A11}$$

Since $S_i > 0$ and $\lambda_j \geq 0$, therefore $c_i \geq 0$. Thus, the min $\|H\|_F$ solution always returns physically implementable solutions.

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